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THE STRESS INTENSITY FACTORS FOR A GRIFFITH CRACK WHOSE SURFACES ARE LOADED ASYMMETRICALLY

I. N. SNEDDON and U. B. C. O. EJIKE*

Department of Mathematics
The University of Glasgow, Glasgow

Abstract—Formulae for the calculation of the stress intensity factors at the tip of a Griffith crack, and for the normal component of the surface displacement, are derived for a crack whose surfaces are subjected to completely arbitrary surface tractions.

1. INTRODUCTION

THE USE of integral transform methods to calculate the distribution of stress in the vicinity of a Griffith crack under a symmetrical pressure is well-known [1, 2]. In this paper we consider the general problem of the loading of a Griffith crack by an unsymmetrical distribution of surface tractions (cf. Fig. 1). We assume that the Griffith crack $|x| \leq a, y = 0$ is opened

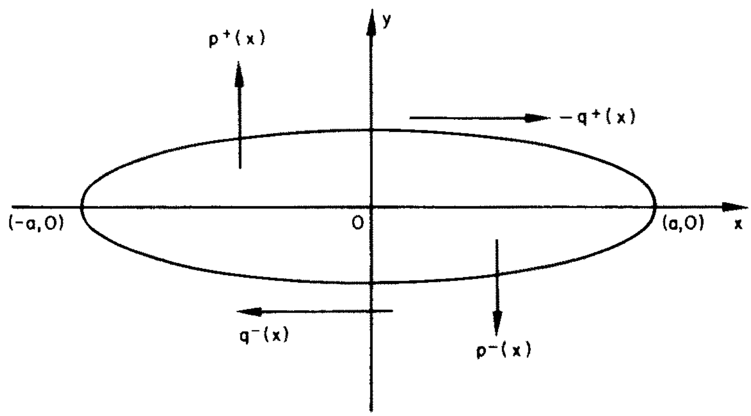

FIG. 1.

up under the action of the forces shown so that on the upper surface of the crack we have the conditions

$$\sigma_{yy}(x, 0+) = -p^+(x), \qquad |x| \leq a, \tag{1.1}$$

$$\sigma_{xy}(x, 0+) = q^+(x) \qquad |x| \leq a, \tag{1.2}$$

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while on the lower surface we have the conditions

$$\sigma_{yy}(x, 0-) = -p^{-}(x), \qquad |x| \leq a, \tag{1.3}$$

$$\sigma_{xy}(x, 0-) = -q^{-}(x), \qquad |x| \leq a. \tag{1.4}$$

We solve the equations of elastic equilibrium for the entire xy-plane with the Griffith crack by considering suitably formulated mixed boundary value problems for the half-planes $y \geq 0$ and $y \leq 0$. In the region of the x-axis outside of the crack we assume that all the components of the displacement vector and of the stress tensor are continuous, i.e. that when $|x| > a$, $y = 0$ we have the conditions

$$u_x(x, 0+) = u_x(x, 0-) \tag{1.5}$$

$$u_y(x, 0+) = u_y(x, 0-) \tag{1.6}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) \tag{1.7}$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-). \tag{1.8}$$

We also have the continuity condition

$$\sigma_{xx}(x, 0+) = \sigma_{xx}(x, 0-), \qquad |x| > a,$$

but it is easily shown that if the conditions (1.5) through (1.8) are satisfied, this last condition is satisfied automatically.

In applications to fracture mechanics of the solution of crack problems in the classical theory of elasticity most interest is centred on the calculation of the stress-intensity factors at the crack tips [3, 4]. These may be defined by the equations

$$K_{+} = \lim_{x \to a+} \sqrt{(x-a)}\sigma_{yy}(x, 0), \tag{1.9}$$

$$K_{-} = \lim_{x \to -a-} \sqrt{(-x-a)}\sigma_{yy}(x, 0), \tag{1.10}$$

$$N_{+} = \lim_{x \to a+} \sqrt{(x - a)}\sigma_{xy}(x, 0), \tag{1.11}$$

$$N_{-} = \lim_{x \to -a-} \sqrt{(-x-a)}\sigma_{xy}(x, 0). \tag{1.12}$$

Even if the main interest is in the computation of the stress-intensity factor it is also important to calculate the normal component of the displacement of the crack faces. The kind of solution used here—which depends on a systematic use of the theory of dual integral equations—remains valid only so long as the inequality

$$u_y(x, 0+) > u_y(x, 0-), \qquad |x| < a, \tag{1.13}$$

is satisfied. As pointed out recently by Burniston [5] it is possible to impose loading conditions which cause crack surfaces to touch in the vicinity of the centre of the crack, which in turn sets up entirely different boundary conditions and renders invalid the type of solution employed here.

In this paper section 2 contains an account of the solution of the general problem by means of Fourier transforms; it is shown that certain arbitrary functions entering into this general solution can be determined from the solutions of four pairs of dual integral equations. These solutions can be found by "elementary methods"; the details are given in section 3. The determinations of the stress-intensity factors and of the shape of the crack are considered, respectively, in sections 4 and 5.

2. GENERAL SOLUTION OF THE EQUATIONS OF EQUILIBRIUM

It is easily shown (cf. for example [2], p. 402) that the equations of elastic equilibrium have solutions of the form

$$\sigma_{xx}(x, y) = \mathrm{D}_x\mathscr{F}_s[\xi^{-2}G_{yy}(\xi, y);\xi \to x]-\mathrm{D}_x\mathscr{F}_c[\xi^{-2}H_{yy}(\xi, y);\xi \to x], \tag{2.1}$$

$$\sigma_{xy}(x, y) = -\mathrm{D}_x\mathscr{F}_c[\xi^{-1}G_y(\xi, y);\xi \to x] - \mathrm{D}_x\mathscr{F}_s[\xi^{-1}H_y(\xi, y);\xi \to x], \tag{2.2}$$

$$\sigma_{yy}(x, y) = -\mathrm{D}_x\mathscr{F}_s[G(\xi, y);\xi \to x]+\mathrm{D}_x\mathscr{F}_c[H(\xi, y);\xi \to x], \tag{2.3}$$

where

$$\mathrm{D}_x = \frac{\partial}{\partial x}$$

and the functions $G(\xi, y)$ and $H(\xi, y)$ both satisfy the equation

$$(\mathrm{D}_y^2-\xi^2)^2\chi(\xi, y) = 0. \tag{2.4}$$

$H_y(\xi, y)$ denotes $\partial H(\xi, y)/\partial y$ etc. As usual $\mathscr{F}_s$ and $\mathscr{F}_c$ denote the operators of the Fourier sine and the Fourier cosine transforms respectively:

$$\mathscr{F}_s[F(\xi, y);\xi \to x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(\xi, y)\sin(\xi x)\,d\xi,$$

$$\mathscr{F}_c[F(\xi, y);\xi \to x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(\xi, y)\cos(\xi x)\,d\xi.$$

The corresponding expressions for the components u_x, u_y of the displacement are given by the pair of equations

$$\begin{aligned}(1+\eta)^{-1}Eu_x(x, y) = {} & \mathscr{F}_s[\xi^{-2}\{(1-\eta)G_{yy}+\eta\xi^2G\};\xi \to x] \\ & -\mathscr{F}_c[\xi^{-2}\{(1-\eta)H_{yy}+\eta\xi^2H\};\xi \to x],\end{aligned} \tag{2.5}$$

$$\begin{aligned}(1+\eta)^{-1}Eu_y(x, y) = {} & \mathscr{F}_c[\xi^{-3}\{(1-\eta)G_{yyy}-(2-\eta)\xi^2G_y\};\xi \to x] \\ & +\mathscr{F}_s[\xi^{-3}\{(1-\eta)H_{yyy}-(2-\eta)\xi^2H_y\};\xi \to x],\end{aligned} \tag{2.6}$$

in which η denotes Poisson's ratio and E the Young's modulus of the material forming the infinite body.

The solutions of equation (2.4) corresponding to the upper half-plane $y \geq 0$ are

$$G(\xi, y) = [A_1(\xi)+\xi y\{A_1(\xi)-B_1(\xi)\}]e^{-\xi y}, \tag{2.7}$$

$$H(\xi, y) = [A_2(\xi)+\xi y\{A_2(\xi)-B_2(\xi)\}]e^{-\xi y}, \tag{2.8}$$

where A_1, A_2, B_1 and B_2 are functions of ξ alone. For these solutions equations (2.5), (2.6), (2.2) and (2.3) give respectively

$$(1+\eta)^{-1}Eu_x(x,0+) = -\mathscr{F}_s[\{(1-2\eta)A_1(\xi)-2(1-\eta)B_1(\xi)\};x] \\ -\mathscr{F}_c[\{(1-2\eta)A_2(\xi)-2(1-\eta)B_2(\xi)\};x] \tag{2.9}$$

$$(1+\eta)^{-1}Eu_y(x,0+) = \mathscr{F}_c[2(1-\eta)A_1(\xi)-(1-2\eta)B_1(\xi);x] \\ +\mathscr{F}_s[2(1-\eta)A_2(\xi)-(1-2\eta)B_2(\xi);x], \tag{2.10}$$

$$\sigma_{xy}(x,0+) = \mathrm{D}_x\mathscr{F}_c[B_1(\xi);x]+\mathrm{D}_x\mathscr{F}_s[B_2(\xi);x], \tag{2.11}$$

$$\sigma_{yy}(x,0+) = -\mathrm{D}_x\mathscr{F}_s[A_1(\xi);x]+\mathrm{D}_x\mathscr{F}_c[A_2(\xi);x]. \tag{2.12}$$

On the other hand, the solutions of (2.4) corresponding to $y \leq 0$ are

$$G(\xi, y) = [A_3(\xi)+\xi y\{B_3(\xi)-A_3(\xi)\}]e^{\xi y}, \tag{2.13}$$

$$H(\xi, y) = [A_4(\xi)+\xi y\{B_4(\xi)-A_4(\xi)\}]e^{\xi y}. \tag{2.14}$$

For these solutions

$$(1+\eta)^{-1}Eu_x(x,0-) = \mathscr{F}_s[\{2(1-\eta)B_3(\xi)-(1-2\eta)A_3(\xi)\};x] \\ -\mathscr{F}_c[\{2(1-\eta)B_4(\xi)-(1-2\eta)A_4(\xi)\};x], \tag{2.15}$$

$$(1+\eta)^{-1}Eu_y(x,0-) = \mathscr{F}_c[\{(1-2\eta)B_3(\xi)-2(1-\eta)A_3(\xi)\};x] \\ +\mathscr{F}_s[\{(1-2\eta)B_4(\xi)-2(1-\eta)A_4(\xi)\};x], \tag{2.16}$$

$$\sigma_{xy}(x,0-) = -\mathrm{D}_x\mathscr{F}_c[B_3(\xi);x]-\mathrm{D}_x\mathscr{F}_s[B_4(\xi);x], \tag{2.17}$$

$$\sigma_{yy}(x,0-) = -\mathrm{D}_x\mathscr{F}_s[A_3(\xi);x]+\mathrm{D}_x\mathscr{F}_c[A_4(\xi);x]. \tag{2.18}$$

The forms of the eight functions $A_j(\xi), B_j(\xi)$, $(j = 1,\ldots,4)$ are determined by the boundary and continuity conditions. In the next section we show how the problem of determining these functions is reduced to that of solving four pairs of dual integral equations.

3. SOLUTION OF THE DUAL INTEGRAL EQUATIONS

Equations (1.1), (2.12) imply the equation

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi);x]-\mathrm{D}_x\mathscr{F}_c[A_2(\xi);x] = p^+(x), \qquad |x| < a.$$

If we denote the odd and even parts of $p^+(x)$ by $p_e^+(x)$ and $p_0^+(x)$ respectively, so that

$$p_e^+(x) = \tfrac{1}{2}[p^+(x)+p^+(-x)], \qquad p_0^+(x) = \tfrac{1}{2}[p^+(x)-p^+(-x)],$$

then this equation can be replaced by the two equations

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi);x] = p_e^+(x), \qquad 0 < x < a, \tag{3.1}$$

$$\mathrm{D}_x\mathscr{F}_c[A_2(\xi);x] = -p_0^+(x), \quad 0 < x < a, \tag{3.2}$$

In a similar way equations (1.2) and (2.11) imply the equations

$$\mathrm{D}_x\mathscr{F}_c[B_1(\xi);x] = q_0^+(x), \qquad 0 < x < a, \tag{3.3}$$

$$\mathrm{D}_x\mathscr{F}_s[B_2(\xi);x] = q_e^+(x), \qquad 0 < x < a, \tag{3.4}$$

equations (1.3) and (2.18) the equations

$$\mathrm{D}_x\mathscr{F}_s[A_3(\xi);x] = p_e^-(x), \qquad 0 < x < a, \tag{3.5}$$

$$\mathrm{D}_x\mathscr{F}_c[A_4(\xi);x] = -p_0^-(x), \qquad 0 < x < a, \tag{3.6}$$

and equations (1.4) and (2.17) the equations

$$\mathrm{D}_x\mathscr{F}_c[B_3(\xi);x] = q_0^-(x), \qquad 0 < x < a, \tag{3.7}$$

$$\mathrm{D}_x\mathscr{F}_s[B_4(\xi);x] = q_e^-(x), \qquad 0 < x < a. \tag{3.8}$$

The equations (1.5), (2.9), (2.15) lead by similar reasoning to the pair of equations

$$(1-2\eta)\mathscr{F}_s[A_1(\xi)-A_3(\xi);x]-2(1-\eta)\mathscr{F}_s[B_1(\xi)-B_3(\xi);x] = 0, \qquad x > a, \tag{3.9}$$

$$(1-2\eta)\mathscr{F}_c[A_2(\xi)-A_4(\xi);x]-2(1-\eta)\mathscr{F}_c[B_2(\xi)-B_4(\xi);x] = 0, \qquad x > a, \tag{3.10}$$

and equations (1.6), (2.10) and (2.16) to the pair

$$2(1-\eta)\mathscr{F}_c[A_1(\xi)+A_3(\xi);x]-(1-2\eta)\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] = 0, \qquad x > a, \tag{3.11}$$

$$2(1-\eta)\mathscr{F}_s[A_2(\xi)+A_4(\xi);x]-(1-2\eta)\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] = 0, \qquad x > a. \tag{3.12}$$

In a similar way we can easily show that equations (1.8), (2.12) and (2.18) are equivalent to the pair of equations

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi)-A_3(\xi);x] = 0 \qquad x > a, \tag{3.13}$$

$$\mathrm{D}_x\mathscr{F}_c[A_2(\xi)-A_4(\xi);x] = 0, \qquad x > a, \tag{3.14}$$

and that equations (1.7), (2.11) and (2.17) are equivalent to the pair

$$\mathrm{D}_x\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] = 0, \qquad x > a, \tag{3.15}$$

$$\mathrm{D}_x\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] = 0, \qquad x > a. \tag{3.16}$$

Integrating both sides of equation (3.15), we find that

$$\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] = b_1, \qquad x > a,$$

where b_1 is a constant. By the Riemann–Lebesgue lemma we see that, provided $B_1(\xi)+B_3(\xi)$ satisfies certain differentiability and integrability conditions, the integral on the left side of this equation tends to zero as $x \to \infty$; hence $b_1 = 0$, i.e. equation (3.15) may be replaced by

$$\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] = 0, \qquad x > a. \tag{3.17}$$

Hence from equation (3.11) we deduce that

$$\mathscr{F}_c[A_1(\xi)+A_3(\xi);x] = 0, \qquad x > a. \tag{3.18}$$

Similarly from equations (3.16) and (3.12) we deduce the pair of equations

$$\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] = 0, \qquad x > a, \tag{3.19}$$

$$\mathscr{F}_s[A_2(\xi)+A_4(\xi);x] = 0, \qquad x > a. \tag{3.20}$$

In a similar way we deduce from equations (3.9) and (3.13) that

$$\mathscr{F}_s[A_1(\xi)-A_3(\xi);x] = 0, \qquad x > a, \tag{3.21}$$

$$\mathscr{F}_s[B_1(\xi)-B_3(\xi);x] = 0, \qquad x > a, \tag{3.22}$$

and from equations (3.10) and (3.14) that

$$\mathscr{F}_c[A_2(\xi)-A_4(\xi); x] = 0, \qquad x > a, \tag{3.23}$$

$$\mathscr{F}_c[B_2(\xi)-B_4(\xi); x] = 0, \qquad x > a. \tag{3.24}$$

Equations (3.1), (3.5) and (3.13) imply that

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi)-A_3(\xi); x] = p_e^+(x)-p_e^-(x), \qquad 0 < x < a,$$

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi)-A_3(\xi); x] = 0, \qquad x > a,$$

so that we have the simple relation

$$A_1(\xi)-A_3(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi^{-1}\int_0^a \{p_e^+(u)-p_e^-(u)\}\cos(\xi u)\,\mathrm{d}u \tag{3.25}$$

as a result of the Fourier inversion theorem.

On the other hand equations (3.1), (3.5) and (3.18) are equivalent to the pair of dual integral equations

$$\mathrm{D}_x\mathscr{F}_s[A_1(\xi)+A_3(\xi); x] = p_e^+(x)+p_e^-(x), \qquad 0 < x < a,$$

$$\mathscr{F}_c[A_1(\xi)+A_3(\xi); x] = 0, \qquad x > a.$$

The solution of this pair of integral equations is elementary (cf. [6]). It may be written in the form

$$A_1(\xi)+A_3(\xi) = \int_0^a tg_1(t)J_0(\xi t)\,\mathrm{d}t, \tag{3.26}$$

where

$$g_1(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^t \{p_e^+(u)+p_e^-(u)\}\frac{\mathrm{d}u}{\sqrt{(t^2-u^2)}}. \tag{3.27}$$

The functions $A_1(\xi)$ and $A_3(\xi)$ can therefore be determined from equations (3.25) and (3.26). The functions $A_2(\xi)$ and $A_4(\xi)$ can be found by a similar method. Equations (3.2), (3.6) and (3.14) imply that

$$\mathrm{D}_x\mathscr{F}_c[A_2(\xi)-A_4(\xi); x] = p_0^-(x)-p_0^+(x), \qquad 0 < x < a,$$

$$\mathrm{D}_x\mathscr{F}_c[A_2(\xi)-A_4(\xi); x] = 0, \qquad x > a,$$

so that

$$A_2(\xi)-A_4(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \xi^{-1}\int_0^a \{p_0^+(u)-p_0^-(u)\}\sin(\xi u)\,\mathrm{d}u, \tag{3.28}$$

whereas equations (3.2), (3.6) and (3.20) imply that $A_2(\xi)+A_4(\xi)$ is the solution of the pair of dual integral equations

$$\mathrm{D}_x\mathscr{F}_c[A_2(\xi)+A_4(\xi); x] = -p_0^+(x)-p_0^-(x), \qquad 0 < x < a,$$

$$\mathscr{F}_s[A_2(\xi)+A_4(\xi); x] = 0, \qquad x > a.$$

The solution of this pair is also elementary. It may be expressed in the form

$$A_2(\xi)+A_4(\xi) = \int_0^a g_2(t)J_1(\xi t)\,\mathrm{d}t, \tag{3.29}$$

where

$$g_2(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^t \{p_0^+(u)+p_0^-(u)\}\frac{u\,\mathrm{d}u}{\sqrt{(t^2-u^2)}}. \tag{3.30}$$

The functions $B_1(\xi)$, $B_2(\xi)$, $B_3(\xi)$ and $B_4(\xi)$ can be determined in a precisely similar fashion. From equations (3.3), (3.7) and (3.15) we deduce that

$$\begin{aligned} \mathrm{D}_x\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] &= q_0^+(x)+q_0^-(x), && 0 < x < a,\\ \mathrm{D}_x\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] &= 0, && x > a, \end{aligned}$$

and hence that

$$B_1(\xi)+B_3(\xi) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\xi^{-1}\int_0^a \{q_0^+(u)+q_0^-(u)\}\sin(\xi u)\,\mathrm{d}u, \tag{3.31}$$

and from equations (3.3), (3.7) and (3.21) that

$$\begin{aligned} \mathrm{D}_x\mathscr{F}_c[B_1(\xi)-B_3(\xi);x] &= q_0^+(x)-q_0^-(x), && 0 < x < a,\\ \mathscr{F}_s[B_1(\xi)-B_3(\xi);x] &= 0, && x > a, \end{aligned}$$

and hence that

$$B_1(\xi)-B_3(\xi) = \int_0^a g_3(t)J_1(\xi t)\,\mathrm{d}t, \tag{3.32}$$

where

$$g_3(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^t \{q_0^-(u)-q_0^+(u)\}\frac{u\,\mathrm{d}u}{\sqrt{(t^2-u^2)}}. \tag{3.33}$$

Finally, from equations (3.4), (3.8) and (3.16) we have that

$$\begin{aligned} \mathrm{D}_x\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] &= q_e^+(x)+q_e^-(x), && 0 < x < a,\\ \mathrm{D}_x\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] &= 0, && x > a, \end{aligned}$$

so that

$$B_2(\xi)+B_4(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\xi^{-1}\int_0^a \{q_e^+(u)+q_e^-(u)\}\cos(\xi u)\,\mathrm{d}u, \tag{3.34}$$

and from equations (3.4), (3.8) and (3.22) that

$$\begin{aligned} \mathrm{D}_x\mathscr{F}_s[B_2(\xi)-B_4(\xi);x] &= q_e^+(x)-q_e^-(x), && 0 < x < a,\\ \mathscr{F}_c[B_2(\xi)-B_4(\xi);x] &= 0, && x > a, \end{aligned}$$

so that

$$B_2(\xi)-B_4(\xi) = \int_0^a t g_4(t)J_0(\xi t)\,\mathrm{d}t, \tag{3.35}$$

where

$$g_4(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^t \{q_e^+(u) - q_e^-(u)\} \frac{du}{\sqrt{(t^2-u^2)}}. \tag{3.36}$$

4. CALCULATION OF THE STRESS-INTENSITY FACTORS

We see from equations (1.9), (1.10) and (2.12) that to calculate the stress intensity factors K_+ and K_- we need to evaluate the two limits

$$k_1 = -\lim_{x \to a+} \sqrt{(x-a)} D_x \mathscr{F}_s[A_1(\xi); x], \tag{4.1}$$

and

$$k_2 = \lim_{x \to a+} \sqrt{(x-a)} D_x \mathscr{F}_c[A_2(\xi); x]. \tag{4.2}$$

Now, from equation (3.26) and an integration by parts, we see that when $x > a$,

$$\mathscr{F}_s[A_1(\xi) + A_3(\xi); x] = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a g_1'(t)\sqrt{(x^2-t^2)}\,dt - \sqrt{\left(\frac{2}{\pi}\right)} g_1(a)\sqrt{(x^2-a^2)},$$

from which we deduce that if $x > a$,

$$D_x \mathscr{F}_s[A_1(\xi) + A_3(\xi); x] = \sqrt{\left(\frac{2}{\pi}\right)} x \int_0^a \frac{g_1'(t)\,dt}{\sqrt{(x^2-t^2)}} - \sqrt{\left(\frac{2}{\pi}\right)} g_1(a) \frac{x}{\sqrt{(x^2-a^2)}}.$$

From equation (3.21) we see that, if $x > a$,

$$D_x \mathscr{F}_s[A_1(\xi); x] = \tfrac{1}{2} D_x \mathscr{F}_s[A_1(\xi) + A_3(\xi); x],$$

so that

$$k_1 = \frac{1}{2}\sqrt{\left(\frac{a}{\pi}\right)} g_1(a). \tag{4.3}$$

Similarly from equation (3.29) we have the relation

$$\mathscr{F}_c[A_2(\xi) + A_4(\xi); x] = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a t^{-1} g_2(t)\,dt + \sqrt{\left(\frac{2}{\pi}\right)} g_2(a) \log\left[\frac{x + \sqrt{(x^2-a^2)}}{a}\right]$$
$$- \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a g'(t) \log\left[\frac{x + \sqrt{(x^2-t^2)}}{t}\right] dt$$

from which we deduce that

$$D_x \mathscr{F}_c[A_2'(\xi) + A_4(\xi); x] = \sqrt{\left(\frac{2}{\pi}\right)} \frac{g_2(a)}{\sqrt{(x^2-a^2)}} - \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \frac{g_2'(t)\,dt}{\sqrt{(x^2-t^2)}}.$$

Using the relation (3.14) we see that

$$k_2 = \frac{1}{2} \frac{g_2(a)}{\sqrt{(\pi a)}}. \tag{4.4}$$

Similarly if we define

$$n_1 = \lim_{x \to a+} \sqrt{(x-a)} \mathrm{D}_x \mathscr{F}_c[B_1(\xi); x] \tag{4.5}$$

$$n_2 = \lim_{x \to a+} \sqrt{(x-a)} \mathrm{D}_x \mathscr{F}_s[B_2(\xi); x] \tag{4.6}$$

we find from equations (3.15) and (3.32) that

$$n_1 = \frac{1}{2} \frac{g_3(a)}{\sqrt{(\pi a)}}, \tag{4.7}$$

and from equations (3.16) and (3.35) that

$$n_2 = \frac{1}{2} \sqrt{\left(\frac{a}{\pi}\right)} g_4(a). \tag{4.8}$$

We therefore have the formulae

$$\begin{aligned} K_+ &= k_1 + k_2, \qquad K_- = k_1 - k_2, \\ N_+ &= n_1 + n_2, \qquad N_- = n_2 - n_1, \end{aligned} \tag{4.9}$$

for the stress-intensity factors in terms of the integrals

$$k_1 = \frac{1}{\pi} (\tfrac{1}{2}a)^{\frac{1}{2}} \int_0^a [p_e^+(x) + p_e^-(x)] \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)}}, \tag{4.10}$$

$$k_2 = \frac{1}{\pi} (2a)^{-\frac{1}{2}} \int_0^a [p_0^+(x) + p_0^-(x)] \frac{x\,\mathrm{d}x}{\sqrt{(a^2 - x^2)}}, \tag{4.11}$$

$$n_1 = \frac{1}{\pi} (2a)^{-\frac{1}{2}} \int_0^a [q_0^-(x) - q_0^+(x)] \frac{x\,\mathrm{d}x}{\sqrt{(a^2 - x^2)}}, \tag{4.12}$$

$$n_2 = \frac{1}{\pi} (\tfrac{1}{2}a)^{\frac{1}{2}} \int_0^a [q_e^+(x) - q_e^-(x)] \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)}}. \tag{4.13}$$

5. THE CALCULATION OF THE CRACK SHAPE

The shape of the crack resulting from the deformation may be calculated by means of equations (2.10) and (2.16).

From these equations we deduce that

$$u_y(x, 0+) - u_y(x, 0-) = \frac{1+\eta}{E} w(x), \qquad |x| \leq a, \tag{5.1}$$

where the function $w(x)$ is defined by the equation

$$\begin{aligned} w(x) = {} & 2(1-\eta)\{\mathscr{F}_c[A_1(\xi) + A_3(\xi); x] + \mathscr{F}_s[A_2(\xi) + A_4(\xi); x] \\ & -(1-2\eta)\{\mathscr{F}_c[B_1(\xi) + B_3(\xi); x] + \mathscr{F}_s[B_2(\xi) + B_4(\xi); x]\}. \end{aligned} \tag{5.2}$$

Now from equations (3.26) and (3.29) we deduce that

$$\mathscr{F}_c[A_1(\xi)+A_3(\xi);x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_x^a \frac{tg_1(t)\,dt}{\sqrt{(t^2-x^2)}}, \qquad 0 < x < a,$$

$$\mathscr{F}_s[A_2(\xi)+A_4(\xi);x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_x^a \frac{xg_2(t)\,dt}{t\sqrt{(t^2-x^2)}}, \qquad 0 < x < a.$$

Integrating both sides of equations (3.3) and (3.7) with respect to x from $x < a$ to a and adding we find that

$$\mathscr{F}_c[B_1(\xi)+B_3(\xi);x] = \mathscr{F}_c[B_1(\xi)+B_3(\xi);a] - \int_x^a \{q_0^+(u)+q_0^-(u)\}\,du.$$

Similarly from equations (3.4) and (3.8) we find that

$$\mathscr{F}_s[B_2(\xi)+B_4(\xi);x] = \mathscr{F}_s[B_2(\xi)+B_4(\xi);a] - \int_x^a \{q_e^+(u)+q_e^-(u)\}\,du.$$

Since the condition $u_y(a,0+) = u_y(a,0-)$ is equivalent to the condition $w(a) = 0$ we deduce that

$$\mathscr{F}_c[B_1(\xi)+B_3(\xi);a] + \mathscr{F}_s[B_2(\xi)+B_4(\xi);a] = 0,$$

and hence that

$$\begin{aligned} w(x) = {} & 2(1-\eta)\sqrt{\left(\frac{2}{\pi}\right)}\int_x^a [tg_1(t)+xt^{-1}g_2(t)]\frac{dt}{\sqrt{(t^2-x^2)}} \\ & +(1-2\eta)\int_x^a [q_e^+(u)+q_0^+(u)+q_e^-(u)+q_0^-(u)]\,du, \qquad 0 < x < a. \end{aligned} \tag{5.3}$$

In a precisely similar way we can show that

$$\begin{aligned} w(x) = {} & 2(1-\eta)\sqrt{\left(\frac{2}{\pi}\right)}\int_{|x|}^a [tg_1(t)-|x|t^{-1}g_2(t)]\frac{dt}{\sqrt{(t^2-x^2)}} \\ & -(1-2\eta)\int_{|x|}^a \{q_e^+(u)+q_e^-(u)-q_0^+(u)-q_0^-(u)\}\,du, \qquad -a < x < 0. \end{aligned} \tag{5.4}$$

The necessary condition (1.13) for the validity of the solution can now be replaced by the condition

$$w(x) > 0, \qquad |x| < a. \tag{5.5}$$

To complete the calculation of the crack shape we need an expression for the function $z(x)$, where

$$u_y(x,0+)+u_y(x,0-) = \frac{1+\eta}{E}z(x). \tag{5.6}$$

From equations (2.10) and (2.16) we deduce that

$$\begin{aligned} z(x) = {} & 2(1-\eta)\{\mathscr{F}_c[A_1(\xi)-A_3(\xi);x] + \mathscr{F}_s[A_2(\xi)-A_4(\xi);x]\} \\ & -(1-2\eta)\{\mathscr{F}_c[B_1(\xi)-B_3(\xi);x] - \mathscr{F}_s[B_2(\xi)-B_4(\xi);x]\}. \end{aligned}$$

The solution (3.32) yields the result

$$\mathscr{F}_c[B_1(\xi)-B_3(\xi);x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^a t^{-1}g_3(t)\,\mathrm{d}t - \sqrt{\left(\frac{2}{\pi}\right)}\int_0^x \frac{xg_3(t)\,\mathrm{d}t}{t\sqrt{(x^2-t^2)}}.$$

and the solution (3.35) gives

$$\mathscr{F}_s[B_2(\xi)-B_4(\xi);x] = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^x \frac{tg_4(t)\,\mathrm{d}t}{\sqrt{(x^2-t^2)}}.$$

The function

$$z_1(x) = \mathscr{F}_c[A_1(\xi)-A_3(\xi);x], \qquad 0 < x < a, \tag{5.7}$$

can be calculated in any particular case by means of equation (3.25) and the function

$$z_2(x) = \mathscr{F}_s[A_2(\xi)-A_4(\xi);x], \qquad 0 < x < a, \tag{5.8}$$

by means of equation (3.28).

We therefore find that the function $z(x)$ is given by the equations

$$\begin{aligned} z(x) &= 2(1-\eta)\{z_1(x)+z_2(x)\} \\ &\quad -(1-2\eta)\sqrt{\left(\frac{2}{\pi}\right)}\left\{\int_0^a t^{-1}g_3(t)\,\mathrm{d}t - \int_0^x [xt^{-1}g_3(t)-tg_4(t)]\frac{\mathrm{d}t}{\sqrt{(x^2-t^2)}}\right\}, \qquad 0 < x < a; \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} z(x) &= 2(1-\eta)\{z_1(|x|)-z_2(|x|)\} \\ &\quad -(1-2\eta)\sqrt{\left(\frac{2}{\pi}\right)}\left\{\int_0^a t^{-1}g_3(t)\,\mathrm{d}t - \int_0^{|x|} [|x|t^{-1}\;g_3(t)+tg_4(t)]\frac{\mathrm{d}t}{\sqrt{(x^2-t^2)}}\right., \\ &\qquad -a < x < 0. \end{aligned} \tag{5.10}$$

REFERENCES

[1] I. N. Sneddon and H. A. Elliott, The opening of a Griffith crack under internal pressure. *Q. appl. Math.* **4**, 262 (1946).

[2] I. N. Sneddon, *Fourier Transforms*. McGraw-Hill (1951).

[3] G. I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture. *Adv. appl. Math.* **7**, 55 (1962).

[4] G. R. Irwin, *Fracture Mechanics*, ONR First Symposium on Naval Structural Mechanics. Pergamon Press (1958).

[5] E. E. Burniston, An example of a partially opened Griffith crack. *Int. Jnl fract. Mech.* In press.

[6] M. Lowengrub, A note on Griffith cracks. *Proc. Edinb. math. Soc.* **15**, 131 (1966).

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Абстракт—Приводятся формулы для определения факторов интенсивности напряжений для самой концевой части трещины Гриффита и для нормального компонента перемещения поверхности. Поверхности трещины находятся под влиянием совершенно произвольных сил связи.